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'Explosive' resonant wave interactions in a three-layer fluid flow

BY A. D. D. CRAIK
Department of Applied Mathematics, University of St Andrews, Fife, Scotland
AND J. A. ADAM
Department of Mathematics, New University of Ulster, Coleraine, N. Ireland

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Cairns (1979) has recently shown how the concept of negative wave energy used in plasma physics can be exploited to obtain insights into the linear and nonlinear instability of certain parallel flows. We here analyse the linear stability and nonlinear three-wave resonance in a three-layer flow with step-wise velocity and density profiles. The results are in complete agreement with Cairns' qualitative predictions. In particular, the existence is confirmed of an 'explosive' instability in which all three waves of a resonant triad grow simultaneously while total wave energy is conserved. Such nonlinear instabilities, previously undetected in fluid flows, may well be important in the ocean and atmosphere.

1. Introduction

In a recent paper, Cairns (1979) has drawn attention to the important role of wave energy in the linear and nonlinear instability of parallel flows. In plasma physics, it is well known that waves may have either positive or negative energy, in the sense that exciting them increases or decreases the total energy of the system as viewed from the chosen frame of reference; and that the nature of the interactions among such waves, both linear and nonlinear, is critically dependent on the signs of the respective wave energies. Since a plasma can support many different types of waves, it is particularly important to be able to identify regions of linear or nonlinear instability *before* carrying out a full and detailed analysis of the problem. An awareness of the role of wave energy provides this insight.

Cairns has drawn attention to the fact that similar ideas are of value in fluid mechanics. He has demonstrated that the concept of negative energy applies to a class of stratified parallel flows and that it affords a satisfying explanation of certain known results in linear stability theory (a fact previously recognized by Landahl (1962) and Benjamin (1960, 1963) in the context of flow over compliant walls). In addition, Cairns has identified the existence in such flows of nonlinear three-wave resonant instabilities in which total wave energy is conserved but *all three* wave amplitudes can grow simultaneously, a phenomenon known in plasma physics as 'explosive instability' since the solutions theoretically attain infinite amplitudes after a finite time. Of course, the weakly nonlinear theory must break down before this singularity is reached.

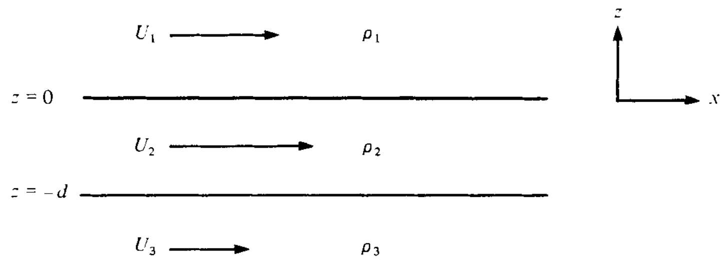

FIGURE 1. The flow configuration.

The theory as outlined by Cairns relates to waves in inviscid parallel flows for which the density profile is piecewise constant and the velocity profile is piecewise constant or piecewise linear. The influence of small viscosity may also be deduced, but extension of the theory to viscous shear flows with curved velocity profiles would entail considerable modification to account for any 'critical layers' in the flow. On the other hand, the theory may readily be modified to deal with continuously-stratified parallel shear flows, in the absence of such critical layers (cf. Acheson 1976).

The general features are demonstrated by Cairns by examination of simple Kelvin–Helmholtz flows. Particularly instructive is the three-layer model, in which a layer of intermediate density ρ_2 is sandwiched between two semi-infinite fluids, with densities ρ_1 and ρ_3, where $\rho_1 \leqslant \rho_2 \leqslant \rho_3$, and each fluid region may have a different uniform velocity, U_i $(i = 1, 2, 3)$. The two interfaces may be subject to both gravity g and surface tension γ_j $(j = 1, 2)$. This configuration is shown in figure 1.

The linear stability of a similar configuration was analysed by Taylor (1931), who noted that instability may occur at flow velocities below those for Kelvin–Helmholtz instability of each interface taken individually. This arises when a wave mode centred on the upper interface and another with the same wavenumber centred on the lower interface have similar frequencies. This instability may be understood, and predicted, without detailed analysis, in terms of the coalescence of positive- and negative-energy wave modes (see Cairns 1979, § 5). The nonlinear stability of this system is also of interest, as it can support resonant triads comprising waves with respective energies of differing signs. When this occurs, the so-called 'explosive instability' takes place, in which all three waves may grow simultaneously, though total wave energy is conserved (see Cairns § 6 who also cites relevant previous work). Although there is a wealth of theoretical, experimental and oceanographic data concerning energy-conserving resonant interactions among three (and four) waves, all of positive energy, there are as yet no experiments or observations which convincingly demonstrate the phenomenon of 'explosive instability' in fluids. This is at least partly due to a lack of awareness that such instability is predicted by theory.

The energy considerations elucidated by Cairns predict that explosive instability occurs in the three-layer configuration, but the actual weakly-nonlinear calculation which is necessary to determine the precise interaction coefficients has not previously been carried out. Some calculations have been done for interactions among internal waves and surface waves (i.e. the three-layer configuration with the density of the upper fluid taken as zero, and no primary flow velocities, $U_i = 0$, $i = 1, 2, 3$), when all waves necessarily have positive energy (Ball 1964; Joyce 1974); but, even for these, the resulting algebraic complexities were sufficient to encourage further approximations. We here present the results of our calculation, without such simplifying approximations, of the weakly nonlinear interaction coefficients for cases of resonant interaction in the three-layer Kelvin–Helmholtz configuration. These calculations confirm the correctness of the predictions based on consideration of the signs of the energies of the respective waves. They further enable estimation of the characteristic growth times of the unstable disturbances.

The present work reveals an important, but overlooked, mechanism of internal wave generation in the ocean and the atmosphere. The presence of a modest current or wind, along with density discontinuities, may be sufficient to cause the energy of an interfacial internal wave (as defined in the chosen reference frame) to become negative. The nonlinear resonant interaction of this negative-energy wave with appropriate positive-energy waves may cause all three waves to grow, at a rate faster than exponential, until limited by higher-order or dissipative effects. This is a potentially more powerful mechanism for generating internal waves than the interactions among three positive-energy waves which have been investigated until now.

2. The linear analysis

The formulation of the problem is basically as in § 5 of Cairns (1979), with the exception that each of the three fluid layers is regarded as having a primary uniform velocity $U_i (i = 1, 2, 3)$, whereas Cairns considers only the uppermost fluid to be in motion. The velocity potential of a general wavelike disturbance satisfies Laplace's equation in each layer and decays to zero as $z \to \pm\infty$: the potential therefore has the form:

$$\left.\begin{aligned} \phi_1 &= C_1 e^{-kz} e^{ikx-i\omega t}, \quad (z > 0); \\ \phi_2 &= (C_2 e^{-kz} + C_3 e^{kz}) e^{ikx-i\omega t}, \quad (-d \leqslant z \leqslant 0); \\ \phi_3 &= C_4 e^{k(z+d)} e^{ikx-i\omega t}, \quad (z < -d); \end{aligned}\right\} \tag{2.1}$$

and the vertical displacements of the interfaces at $z = 0$ and $z = -d$ are

$$\eta_1 = A_1 e^{ikx-i\omega t}, \quad \eta_2 = A_2 e^{ikx-i\omega t}. \tag{2.2}$$

Here, the wavenumber k is taken to be real and positive, and physical quantities correspond to the real parts of the appropriate complex expressions. The kinematic and pressure boundary conditions at the two interfaces lead directly to the linear dispersion relation for the frequency ω,

$$D(\omega, k) \equiv -D_1(\omega, k) + \frac{\Lambda^2(\omega, k)}{D_2(\omega, k)} = 0, \tag{2.3}$$

where

$$D_1(\omega, k) \equiv (\rho_2 - \rho_1) g + \gamma_1 k^2 - k^{-1}\{\rho_1 (kU_1 - \omega)^2 + \rho_2 (kU_2 - \omega)^2 \coth (kd)\},$$

$$D_2(\omega, k) \equiv (\rho_3 - \rho_2) g + \gamma_2 k^2 - k^{-1}\{\rho_3 (kU_3 - \omega)^2 + \rho_2 (kU_2 - \omega)^2 \coth (kd)\}$$

and

$$\Lambda(\omega, k) \equiv \rho_2 k^{-1} (kU_2 - \omega)^2 \operatorname{cosech} (kd).$$

The coefficients of surface tension at the upper and lower interfaces are denoted by γ_1, γ_2 respectively. Note that, as in Cairns' § 5, the dispersion relation for waves on the upper interface if the lower one is replaced by a rigid boundary at $z = -d$ is just

$D_1(\omega, k) = 0$. Similarly, $D_2(\omega, k) = 0$ is the dispersion relation for waves on the lower interface when the upper interface $z = 0$ is rigid. When the quantity Λ is small (i.e. $kd \gg 1$) the coupled system behaves rather like the uncoupled ones, having $D_1(\omega, k)$ and $D_2(\omega, k)$ approximately equal to zero for 'upper' and 'lower' modes respectively. However, throughout this paper, the exact dispersion relation (2.3) is used, and no assumption is made regarding the size of kd.

Clearly, the dispersion relation may be written in various forms; we emphasize that the present form is that resulting from the procedure outlined by Cairns § 2, in which $D(\omega, k)$ represents the difference in pressure (after accounting for surface tension) either just above or just below the interface centred at $z = 0$. The physically admissible disturbances are just those for which this pressure-difference is zero. It is only when the dispersion function $D(\omega, k)$ describes this pressure difference (at any chosen reference depth, not necessarily $z = 0$) that it arises in a physically natural fashion in the expression for the wave energy $E(k, \omega)$ as (see Cairns, § 2)

$$E(k, \omega) = \tfrac{1}{4}\omega \frac{\partial D}{\partial \omega} |A|^2 \tag{2.4}$$

where $|A|$ is the amplitude of the vertical displacement at the chosen reference depth and $\omega \partial D / \partial \omega$ is evaluated at the appropriate root of the dispersion relation $D(\omega, k) = 0$. Since the reference depth is here chosen at $z = 0$, A equals A_1 from (2.2) and $D(\omega, k)$ is as given in (2.3). (Had $z = -d$ been chosen, A would equal A_2 and $D(\omega, k)$ would be as in (2.3) but with D_1 and D_2 interchanged.)

Since equation (2.3) may be re-expressed as a quartic for the unknown frequency ω, the roots are easily computed using standard procedures. These roots are shown in figures 2(a)–(e) for the following illustrative cases:

$$\rho_1 = 1{\cdot}015, \quad \rho_2 = 1{\cdot}020, \quad \rho_3 = 1{\cdot}026\,(\mathrm{g\,cm^{-3}});$$
$$\gamma_1 = 74, \quad \gamma_2 = 0\,(\mathrm{g\,s^{-2}}); \quad g = 981\,(\mathrm{cm\,s^{-2}});$$
$$\text{various } U_1; \quad U_2 = U_3 = 0\,(\mathrm{cm\,s^{-1}}), \quad d = 8\,(\mathrm{cm}).$$

Figure 2(a) shows the four wave modes with no basic flow. For the most part, those labelled 1 and 2 are centred on the lower interface and those labelled 3 and 4 are centred on the upper interface. The exceptions are waves with $kd \leqslant O(1)$, which occur near the origin of figure 2; no such simple identification is then possible since the variation in amplitude over the depth d is small in such cases. Provided kd is not too small, modes 1 and 2 have $D_2 \approx 0$ and modes 3 and 4 have $D_1 \approx 0$.

Figure 2(b) shows the case $U_1 = 3\,\mathrm{cm\,s^{-1}}$. Modes 1 and 2 are virtually unchanged but 3 and 4 are displaced upwards. Had modes 2 and 3 been represented by the approximations $D_2 = 0$ and $D_1 = 0$ respectively, the curves so obtained would intersect near $k = 0{\cdot}35$. But near such an intersection the coupling between the interfaces becomes significant; and, since the energy of both these modes is positive, their interaction is of the passive sort giving no instability but causing an interchange of identity on the continuous curves (see Cairns, figure 3b). This feature is also shown in figure 2(c), where now the energy of mode 3 is on the point of changing sign in the vicinity of $k = 0{\cdot}25\,\mathrm{cm^{-1}}$. In figure 2(d), mode 3 has negative energy wherever it has positive frequency, i.e. for $0{\cdot}1 < k < 0{\cdot}8$ approximately. In 2(e), mode 3 and mode 1 'intersect', and since they have respective negative and positive energies near the intersections,

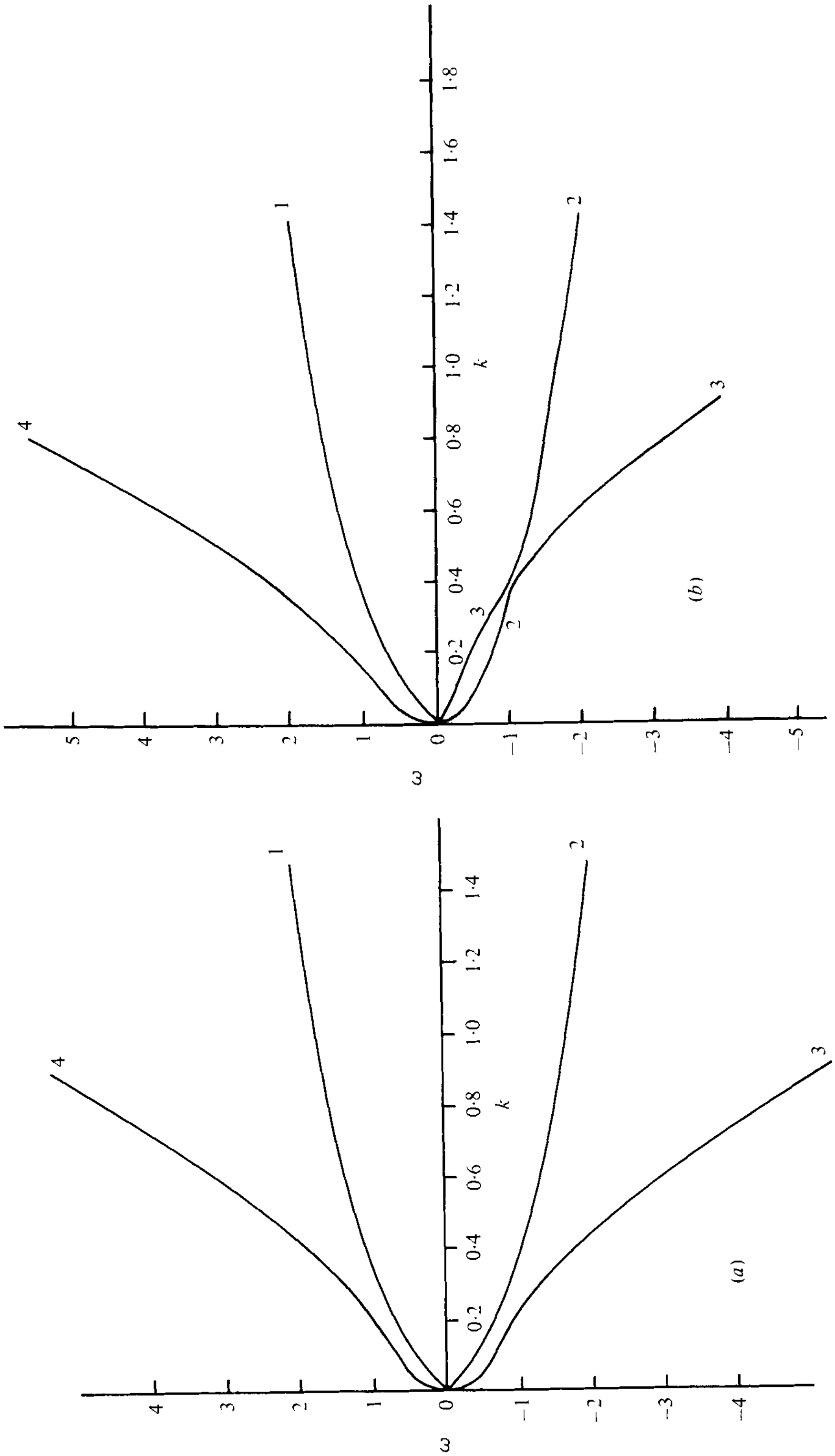

FIGURE 2. For legend see p. 21.

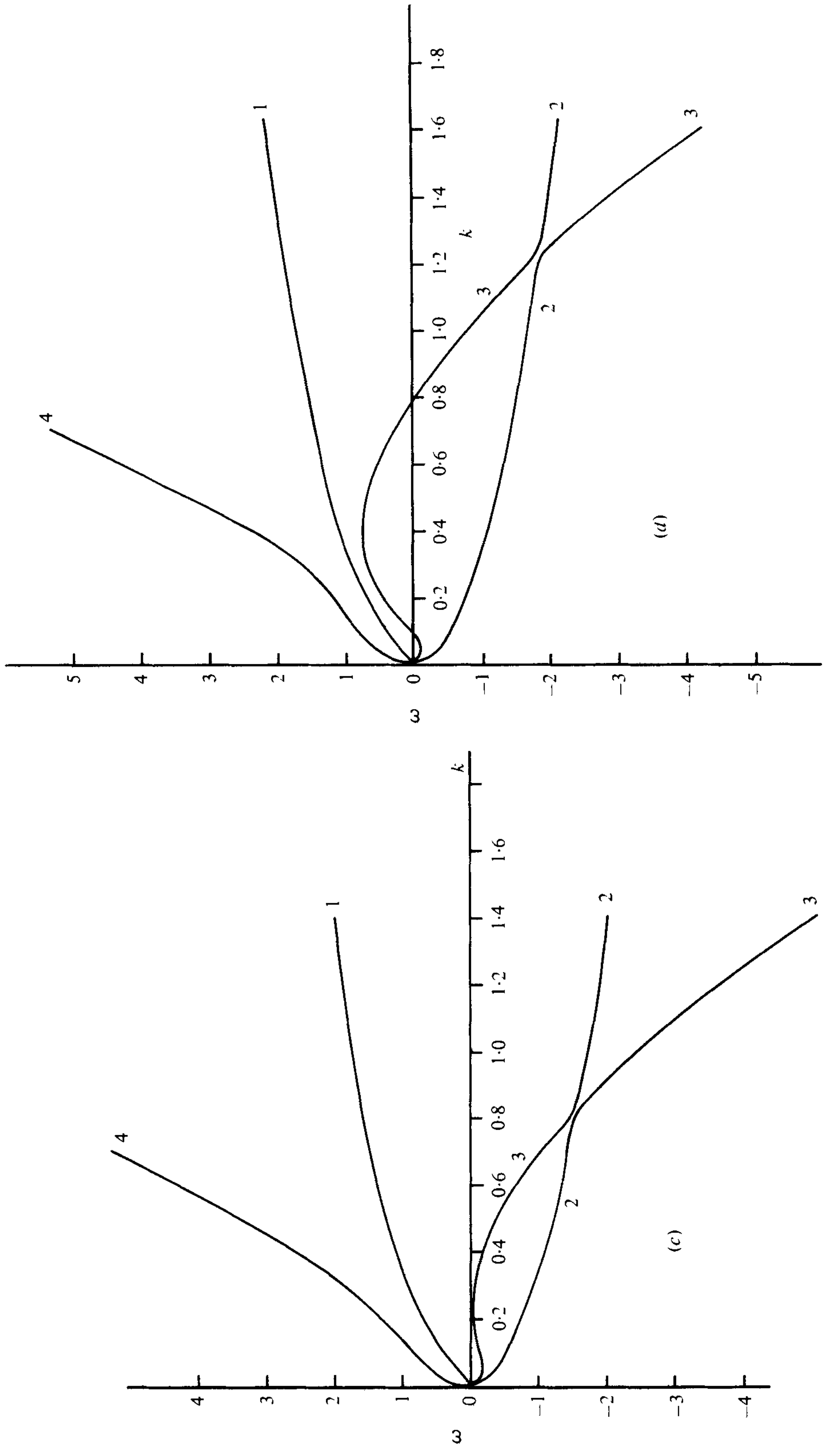

FIGURE 2. For legend see facing page.

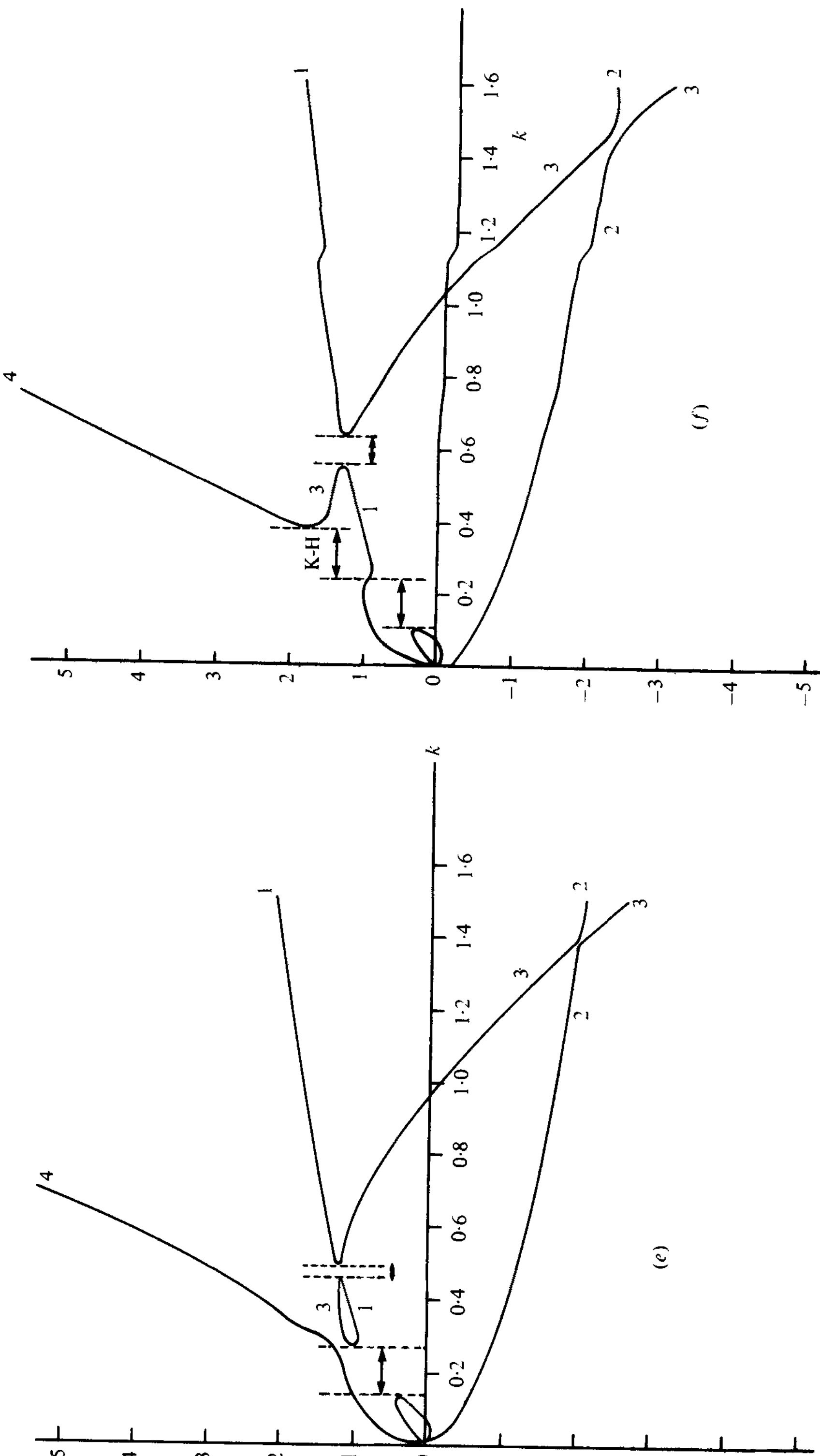

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FIGURE 2. Dispersion curves for various values of U_1 (cm sec^{-1}), see equation (2.3). (*a*) $U_1 = 0$; (*b*) $U_1 = 3{\cdot}0$; (*c*) $U_1 = 6{\cdot}0$; (*d*) $U_1 = 8{\cdot}0$; (*e*) $U_1 = 8{\cdot}7$; (*f*) $U_1 = 9{\cdot}0$.

the coupling between them is of the unstable sort (Cairns, figure 3*c*), giving complex frequencies in the neighbourhoods of $k = 0{\cdot}2$ and $k = 0{\cdot}5\,\text{cm}^{-1}$. In figure 2(*f*), the negative-energy mode 3 and the positive energy mode 4 have 'intersected', giving the classical Kelvin–Helmholtz instability of the upper interface, in addition to the instabilities associated with the coupling of modes 1 and 3. These three unstable regions are shown by double arrows on the diagram, the Kelvin–Helmholtz instability being labelled 'K–H'.

It should be emphasized that the qualitative features of the exact dispersion curves shown in figure 2 can be (and were) successfully predicted from consideration of the *uncoupled* dispersion relations $D_1 = 0$ and $D_2 = 0$ and the energies of the respective uncoupled wave modes. This would be a great advantage in dealing with multi-layer configurations for which the precise coupled dispersion relation is less readily obtained.

3. Resonant triads

'Explosive' instability occurs when three waves are in resonance and the wave of greatest frequency has energy of different sign to the energies of the other two waves (see Cairns § 5). The resonance conditions for three waves with wavenumbers k_1, k_2, k_3 (all greater than zero) and corresponding real frequencies ω_1, ω_2, ω_3 are

$$k_1 = k_2 + k_3, \quad \omega_1 = \omega_2 + \omega_3, \tag{3.1}$$

and the interaction of such waves is of the explosive sort when the wave (k_1, ω_1) has energy of different sign to that of the waves (k_2, ω_2) and (k_3, ω_3). (We here restrict attention to two-dimensional waves; but extension to waves propagating in different directions is quite straightforward.) Investigation of the dispersion diagrams such as those shown in figures 2(*a*)–(*f*) reveals that there exist resonant triads of this kind. In the first instance, this is most easily seen (e.g. Ball 1964) by redrawing the dispersion curves on a transparent sheet then moving the origin of the redrawn curves along one of the original curves (keeping the orientation of the axes constant). A typical example is shown in figure 3. If the origin of the redrawn curves is made to follow the original negative-energy branch 3, it is found that the negative-energy branch on the transparency intersects the positive-energy branch 1 whenever the origin at (k_2, ω_2) lies between the points (0·18, 0·3) and (0·64, 0·8) of branch 3. A point of intersection determines the positive-energy wave (k_1, ω_1), on branch 1 and the negative-energy wave (k_3, ω_3), on the redrawn branch 3, which form a resonant triad with (k_2, ω_2). Once the approximate range of such triads is determined in this way, further computations can be undertaken, using an iterative scheme, to determine the precise values of wavenumber and frequency of particular triads; such results are presented in § 5.

We note, in particular, that 'explosively' resonant triads in cases corresponding to figure 3 comprise two waves of negative energy and one of positive energy; and that, since the configuration of figure 3 is not *linearly* unstable, the explosive instability of these resonant triads provides a *nonlinear* instability mechanism at flow velocities below that for any linear one.

In contrast, figure 4 shows the existence of linear instability in two narrow bands of wavenumber centred approximately where *uncoupled* branches 1 and 3 (with

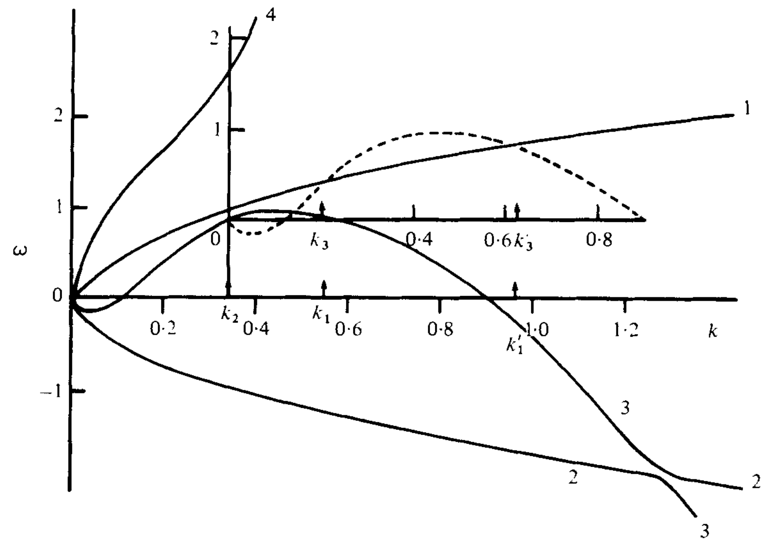

FIGURE 3. Location of 'explosive' resonant triads. The particular case shown is for $\rho_1 = 1{\cdot}010$, $\rho_2 = 1{\cdot}020$, $\rho_3 = 1{\cdot}026\,\mathrm{g\,cm^{-3}}$; $\gamma_1 = 100$, $\gamma_2 = 0\,\mathrm{g\,s^{-2}}$; $g = 981\,\mathrm{cm\,s^{-2}}$, $d = 8\,\mathrm{cm}$, $U_1 = 10$, $U_2 = U_3 = 0\,\mathrm{cm\,s^{-1}}$.

$D_2 = 0$ and $D_1 = 0$) would have intersected. By the method outlined above, it is also found that there are resonant triads comprising two positive-energy waves (on branch 1) and one negative-energy wave: the negative-energy wave (k_1, ω_1) in such cases lies on parts of branch 3 which lie above branch 1. In addition, there are triads of two negative-energy waves and one positive-energy wave for which the positive-energy wave (k_1, ω_1) lies on a portion of branch 1 situated above branch 3. Both these types of triad are of the 'explosive' sort.

It is perhaps worth mentioning that the linear instability of this configuration may be interpreted as a sort of 'degenerate explosive instability'. For, when (uncoupled) positive and negative modes intersect, the intersection point is a point of resonance between the positive and negative modes, and a 'wave' of zero frequency and wavenumber; the resonance is of the explosive sort since, if the 'wave of zero frequency' has positive energy, the negative-energy mode may be considered to be of infinitesimally greater wavenumber than the positive-energy mode (or *vice versa* if the wave of zero frequency has negative energy). In contrast, although the intersection of two linear modes of *like* energy form a resonant triad with a zero-frequency mode, their interaction is of the non-explosive sort, and so is associated not with instability, but with slight changes in frequency from those given by the uncoupled dispersion relations, in agreement with the qualitative curves shown in Cairns' figure 3.

In addition to the 'explosive' triads already mentioned, there are several categories of resonant triads which are not of the 'explosive' kind. For instance, it may be inferred from figure 3 that there are three resonant positive-energy waves, one on each of branches 1, 2, and 4, with that on branch 1 having the largest wavenumber. Also, from figure 3, there are triads of two positive-energy waves and one negative-energy wave, with one wave on each branch 1, 3 and 4: this is 'non-explosive' since the

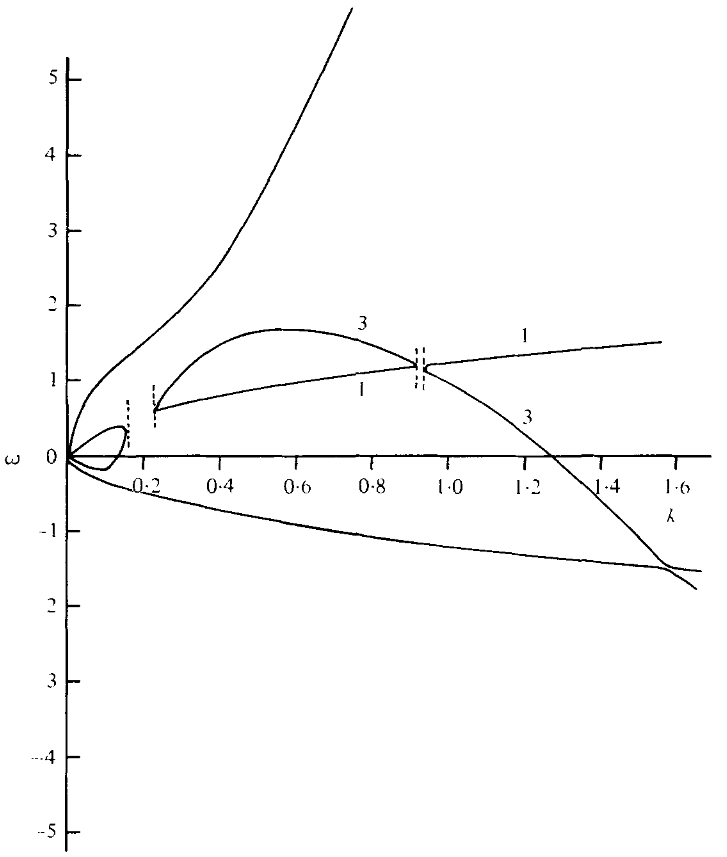

FIGURE 4. Dispersion curve for $\rho_1 = 1{\cdot}010$, $\rho_2 = 1{\cdot}020$, $\rho_3 = 1{\cdot}023\,\mathrm{g\,cm^{-3}}$; $\gamma_1 = 74$, $\gamma_2 = 0\,\mathrm{g\,cm^{-2}}$; $g = 981\,\mathrm{cm\,s^{-2}}$, $d = 8\,\mathrm{cm}$; $U_1 = 10{\cdot}0$, $U_2 = U_3 = 0\,\mathrm{cm\,s^{-1}}$. Resonant triads of modes 1 and 3 are of 'explosive' sort.

negative-energy wave does not have the greatest frequency. There are also resonant triads of three negative-energy waves on branch 3 and three positive-energy waves on branch 4, the latter corresponding, in the case $U_1 = 0$, to the well-known resonance of gravity-capillary waves on a single interface.

It is clear that, even in this simple flow configuration, there is a rich structure of resonant interactions. Energy concepts enable one to tell, without detailed calculation, which of these interactions are of the explosively unstable sort, and which are not. But a precise description of the interaction requires detailed nonlinear analysis.

4. The nonlinear analysis

The means of calculating the evolution equations for the amplitudes of resonantly-interacting weakly-nonlinear waves is well established in principle, but cumbersome and laborious in practice. For resonant interactions of surface gravity-capillary waves in fluid of infinite depth, this analysis has been carried out by McGoldrick (1965), Simmons (1969) and Case & Chiu (1977). Simmons' variational formulation of the problem somewhat reduces the algebraic manipulations, as compared with a direct attack on the nonlinear boundary conditions, and this approach also emphasizes the inherent symmetry of the problem by drawing attention to the interrelationship of the three nonlinear coupling coefficients. Despite McGoldrick's apparent difficulty in establishing analytically that energy is conserved, it is easy, for two-dimensional waves at any rate, to confirm that this is so, working from his equations (3.13) and noting a misprint ($4\sigma_3$ for $4\sigma_1$) in the final equation.

Some work on a three-layer medium but without a primary flow, was done by Simmons in a doctoral dissertation, reported briefly in Simmons (1969). Also, Ball (1964) has considered the resonant interaction of surface and internal waves in a two-layer system using a shallow-water approximation and Thorpe (1966), Joyce (1974) and Watson, West & Cohen (1976) have examined similar interactions for a continuously stratified fluid. Also of interest is the work of Gargett & Hughes (1972) on the interaction of long internal waves and short surface gravity waves; of Djordjevic & Redekopp (1977) on long and short surface wave interaction; and of Craik (1968) on resonant surface gravity waves on a uniform shear flow.

We here present a complete analysis of resonant wave triads for the general inviscid three-layer configuration of figure 1, the only assumption being the usual one of *weak* nonlinearity. The algebraic complexities are formidable, requiring care and persistence, and the results were checked in several ways which are described later.

Each of three waves, with periodicities of the form $\exp[i(k_j - \omega_j t)]$, ($j = 1, 2, 3$) and wavenumbers and frequencies k_j, ω_j satisfying the resonance conditions (3.1), may be represented, to linear approximation, as in § 2, the various functions and constants ϕ_l, C_m, A_n being supplied with superscripts, e.g. $A_n^{(j)}$, where $j = 1, 2, 3$ distinguishes the respective waves. The resonance conditions (3.1) imply a suitable choice of k_1, k_2, k_3 as indicated in the previous section. In addition, a second-order correction must be added to the linear description of each wave, and this is done by the inclusion of further terms, similar to those of (2.1) and (2.2), but with different constants identified by the symbol '^', namely, $\hat{C}_m^{(j)}$ ($m = 1, 2, 3, 4$) and $\hat{A}_n^{(j)}$ ($n = 1, 2$). We recall that the stream function must satisfy Laplace's equation in each fluid layer, and that nonlinearities enter the problem only through the interfacial boundary conditions. It is unnecessary to calculate various second-order but non-resonant terms, such as the second harmonics in $\exp[i(2k_j x - 2\omega_j t)]$, since these play no part in the nonlinear evolution equations at the level of approximation sought. However we note that the omission of such terms excludes the special case where a wave and its own second harmonic form a resonant triad of the form $k_1 = 2k_2$, $\omega_1 = 2\omega_2$. Such cases have been investigated by McGoldrick (1970) and Nayfeh (1973) for waves on a single interface.

The linear constants $C_m^{(j)}$ ($m = 1, 2, 3, 4$) and $A_n^{(j)}$ ($n = 1, 2$) are related by the kinematic and pressure boundary conditions at the two interfaces. In practice, it proved convenient to represent the $C_m^{(j)}$ and $A_2^{(j)}$ in terms of $A_1^{(j)}$, the wave amplitudes at the upper interface. Similarly, the second-order coefficients $\hat{C}_m^{(j)}$ and $\hat{A}_n^{(j)}$ are related to each other and to the linear amplitude functions $A_1^{(j)}(t)$ by the same boundary conditions, taken to second order in wave amplitudes. In doing so, allowance must be made for the slow temporal variations in wave amplitude by retaining terms in $dA_1^{(j)}/dt$. If required, slow spatial variations may also be included without undue difficulty (cf. Simmons, 1969), but these are here omitted, for brevity.

Derivation of the nonlinear kinematic and pressure boundary conditions at the two interfaces is tedious but straightforward. Since this is a well-established procedure, and since the resultant equations are lengthy, the details are omitted. Readers not familiar with this procedure are referred to the papers of McGoldrick (1965, 1970). Essentially, at each interface and for each value of $j = 1, 2, 3$, there are two kinematic conditions (one for each side of the interface) and one pressure condition to be met. These six conditions yield six non-homogeneous equations which are linear in the six unknown constants $\hat{C}_m^{(j)}$ $(m = 1, 2, 3, 4)$ and $\hat{A}_n^{(j)}$ $(n = 1, 2)$ and the unknown time-derivative $\mathrm{d}A_1^{(j)}/dt$, and which contain nonlinear terms, quadratic in the linear wave amplitudes $A_1^{(s)}$ $(s = 1, 2, 3)$, which result from the wave interactions at second order. The form of the linear terms requires a condition for compatibility of these equations, in order that solutions $\hat{C}_m^{(j)}$, $\hat{A}_n^{(j)}$ may exist, and this uniquely determines the growth rate $dA_1^{(j)}/dt$. In practice, it proves convenient to eliminate each of the $\hat{C}_m^{(j)}$ from these equations, yielding just two inhomogeneous equations linear in $\hat{A}_1^{(j)}$ and $\hat{A}_2^{(j)}$. These are of the form

$$\left.\begin{aligned} D_1(\omega_j, k_j)\,\hat{A}_1^{(j)} + \Lambda(\omega_j, k_j)\,\hat{A}_2^{(j)} &= P\,dA_1^{(j)}/dt + R,\\ \Lambda(\omega_j, k_j)\,\hat{A}_1^{(j)} + D_2(\omega_j, k_j)\,\hat{A}_2^{(j)} &= Q\,dA_1^{(j)}/dt + S,\end{aligned}\right\} \tag{4.1}$$

where P, Q are known functions of ω_j and k_j only and R, S represent the nonlinear interaction terms. Because of the linear dispersion relation (2.3), it is clearly necessary that

$$\frac{dA_1^{(j)}}{dt}\left(\frac{P}{D_1} - \frac{Q}{\Lambda}\right) = \frac{S}{\Lambda} - \frac{R}{D_1}. \tag{4.2}$$

This, in schematic form, is the evolution equation for the jth wave. The actual equations, are as follows, where $A_1^{(j)}$ are the complex wave amplitudes at the interface $z = 0$ and * denotes complex conjugation:

$$\left.\begin{aligned} -\frac{dA_1^{(1)}}{dt}\,i\,\frac{\partial D(\omega_1, k_1)}{\partial \omega_1} &= \tfrac{1}{2}\lambda A_1^{(2)} A_1^{(3)},\\ -\frac{dA_1^{(2)}}{dt}\,i\,\frac{\partial D(\omega_2, k_2)}{\partial \omega_2} &= \tfrac{1}{2}\lambda A_1^{(1)} A_1^{(3)*},\\ \text{and}\qquad -\frac{dA_1^{(3)}}{dt}\,i\,\frac{\partial D(\omega_3, k_3)}{\partial \omega_3} &= \tfrac{1}{2}\lambda A_1^{(1)} A_1^{(2)*};\end{aligned}\right\} \tag{4.3}$$

where

$$\left.\begin{aligned} \lambda &= 2\rho_1(U_1 k_2 - \omega_2)(U_1 k_3 - \omega_3) + \rho_2[\mathscr{E} - \mathscr{D}_1(1)\mathscr{D}_1(2) - \mathscr{D}_1(2)\mathscr{D}_1(3) - \mathscr{D}_1(3)\mathscr{D}_1(1)]\\ &\quad + \Theta\{2\rho_3(U_3 k_2 - \omega_2)(U_3 k_3 - \omega_3)\\ &\quad + \rho_2[\mathscr{E} - \mathscr{D}_2(1)\mathscr{D}_2(2) - \mathscr{D}_2(2)\mathscr{D}_2(3) - \mathscr{D}_2(3)\mathscr{D}_2(1)]\},\\ \mathscr{E} &\equiv (\omega_2 - U_2 k_2)^2 + (\omega_2 - U_2 k_2)(\omega_3 - U_2 k_3) + (\omega_3 - U_2 k_3)^2,\\ \Theta &\equiv \frac{D_1(\omega_1, k_1)\,D_1(\omega_2, k_2)\,D_1(\omega_3, k_3)}{\Lambda(\omega_1, k_1)\,\Lambda(\omega_2, k_2)\Lambda(\omega_3, k_3)} = \frac{\Lambda(\omega_1, k_1)\,\Lambda(\omega_2, k_2)\,\Lambda(\omega_3, k_3)}{D_2(\omega_1, k_1)\,D_2(\omega_2, k_2)\,D_2(\omega_3, k_3)},\\ \mathscr{D}_1(j) &\equiv \frac{k_j}{\rho_2(\omega_j - U_2 k_j)}[(\rho_2 - \rho_1)g + \gamma_1 k_j^2 - \rho_1 k_j^{-1}(k_j U_1 - \omega_j)^2],\\ \text{and}\qquad \mathscr{D}_2(j) &\equiv \frac{k_j}{\rho_2(\omega_j - U_2 k_j)}[(\rho_3 - \rho_2)g + \gamma_2 k_j^2 - \rho_3 k_j^{-1}(k_j U_3 - \omega_j)^2].\end{aligned}\right\} \tag{4.4}$$

$D(\omega_j, k_j)$, $D_1(\omega_j, k_j)$, $D_2(\omega_j, k_j)$ and $\Lambda(\omega_j, k_j)$ are as defined in § 2, and j may take the values 1, 2 or 3. Throughout the analysis, use was made of the resonant conditions (3.1).

Several useful checks were employed to detect and eliminate algebraic errors: the left-hand sides of (4.3) had to reduce to the forms shown, proportional to $\partial D/\partial \omega_j$, and the three interaction coefficients, denoted here by λ but in fact calculated separately, had to be identical. [Perhaps unwisely, we did not employ the variational method of Simmons (1969) but dealt directly with the interfacial conditions.] Further, the present result can be shown to reduce to the results of McGoldrick (1965), Simmons (1969) and Case & Chiu (1977) when only one interface (at either $z = 0$ or $z = -d$) is present. We are therefore confident of the correctness of our result.

The energy E_j associated with each wave is (see Cairns 1979, §§ 3–5)

$$E_j = \tfrac{1}{4}\omega_j \frac{\partial D(\omega_j, k_j)}{\partial \omega_j} |A_1^{(j)}|^2$$

and equations (4.3) lead immediately to the energy conservation law

$$(d/dt)\,[E_1 + E_2 + E_3] = \tfrac{1}{4}(\omega_2 + \omega_3 - \omega_1)\,\mathrm{Re}\,\{i\lambda A_2 A_3 A_1^*\} = 0.$$

As already mentioned, the interaction is of the 'explosive' sort whenever

$$\mathrm{sgn}\,(E_1) = \mathrm{sgn}\,(-E_2) = \mathrm{sgn}\,(-E_3).$$

5. Some numerical results

A comprehensive analysis of all possible resonant triads would be a formidable, and rather pointless, task. We here describe some typical results for the 'explosive' type of resonance exemplified by the foregoing discussion of figure 3, in which case there is no linear instability. For each choice of the wavenumber k_2 on the negative-energy branch 3 there are either no resonant triads or two, with k_3 on branch 3 and $k_1 = k_2 + k_3$ on the positive-energy branch 1. Such triads are shown in figure 5 as plots of k_2 versus k_3 for various fixed values of U, the value of k_1 being $k_2 + k_3$. The outermost closed curve for $U_1 = 10{\cdot}0$ corresponds to the dispersion curves of figure 3, and the other curves are for slightly smaller values of U_1 but with all other quantities fixed as in figure 3. This particular resonance disappears at values of U_1 below about $9{\cdot}6\ \mathrm{cm\,s^{-1}}$. Naturally, these curves are symmetric about the line $k_2 = k_3$.

Corresponding to the closed curve $U_1 = 10{\cdot}0$ in figure 5, the coupling coefficient λ has been computed from (4.4) as a function of k_2. As one traverses the lower-right half of the curve $U_1 = 10{\cdot}0$ of figure 5 from A to C, k_2 varies from $0{\cdot}244$ at A to $0{\cdot}64$ at B and back to $0{\cdot}57$ at C, while k_3 assumes the appropriate value denoted by the curve and k_1 equals $k_2 + k_3$. The upper-left half of the curve of course yields the same values of λ. Figure 6 shows these values. It should be noted that there is one exceptional case ($k_2 \approx 0{\cdot}35$, $k_3 \approx 0{\cdot}19$) for which the interaction coefficient λ is precisely *zero* and hence no resonant instability can occur. For all other non-zero values of λ, the resonance is of the explosive sort, as already mentioned. Such exceptional cases cannot be predicted by energy considerations. [A case in which λ is zero for a whole class of triads satisfying the resonance conditions is briefly reported by Simmons (1969, § 7): this was explained by symmetry arguments.]

To complete the calculation of the interaction equations (4.3) it is necessary to

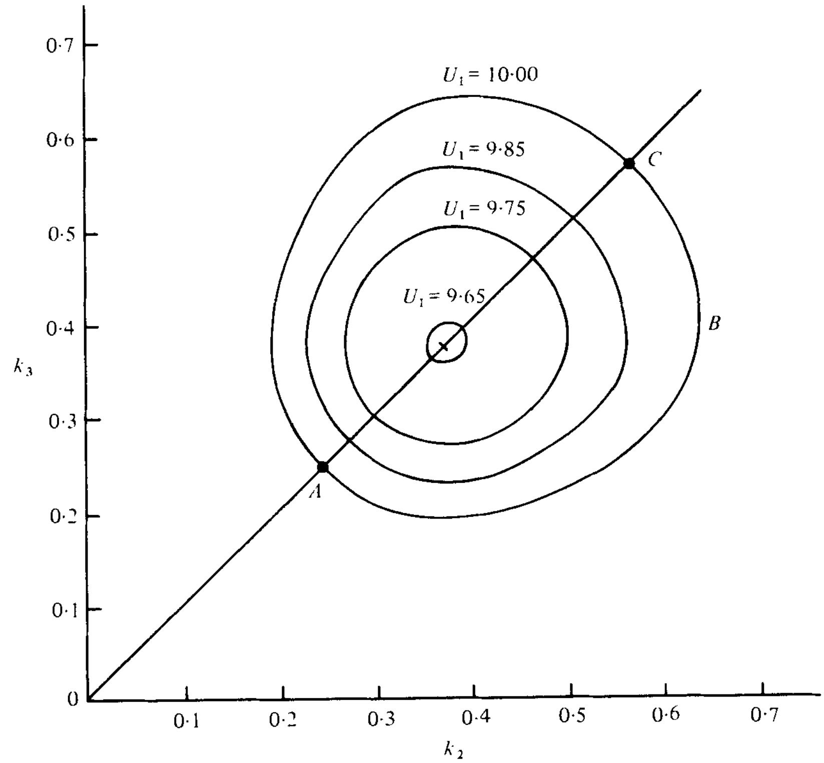

FIGURE 5. Resonance curves, k_2 *vs.* k_3 (and $k_1 = k_2 + k_3$) for various U_1. Other data are as in figure 3.

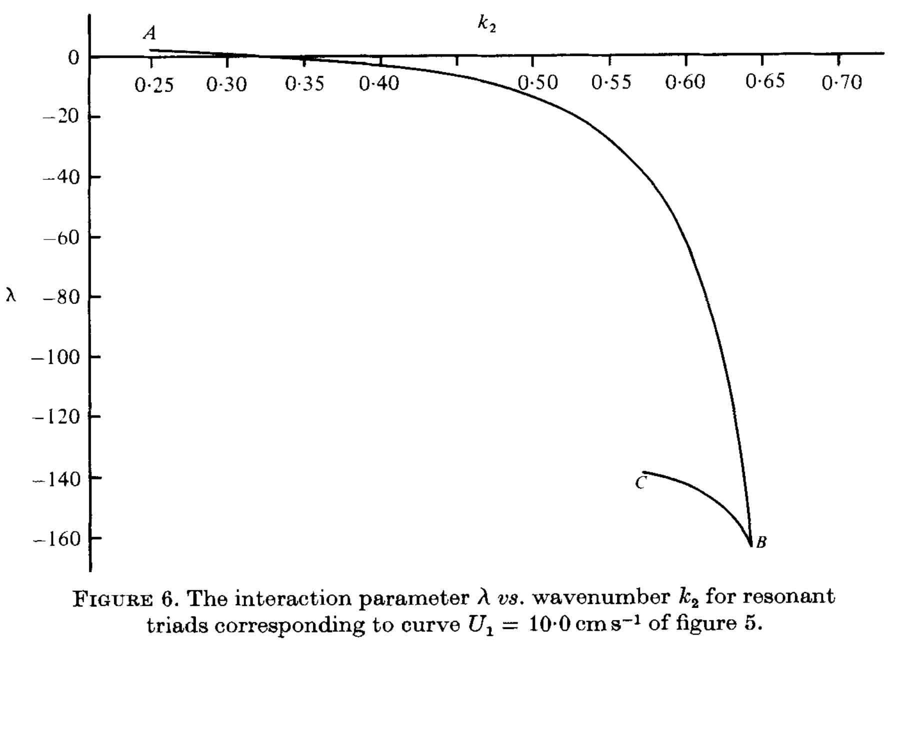

FIGURE 6. The interaction parameter λ *vs.* wavenumber k_2 for resonant triads corresponding to curve $U_1 = 10{\cdot}0\,\mathrm{cm\,s^{-1}}$ of figure 5.

evaluate the quantities $\partial D(\omega_j, k_j)/\partial \omega_j$ on the left-hand sides. This was done using result (2.3) together with the appropriate resonance conditions, for the same case $U_1 = 10{\cdot}0$ as above. These are shown in figures 7 (a, b), where we have chosen to plot $\omega \partial D/\partial \omega$ versus k, since this quantity is proportional to wave energy. Figure 7 (a) shows results for the waves centred on the upper interface, corresponding to mode 3; this clearly demonstrates that the energy is negative for $0{\cdot}11 < k < 0{\cdot}91$ approximately, in agreement with figure 3. In figure 7 (b), which describes the positive-energy mode 1 centred at the lower interface, a logarithmic scale has been used. This is necessary since the amplitudes $A_1^{(j)}$ are those measured at the *upper* interface. For mode 1, this amplitude is smaller by a factor of order $O(e^{-kd})$ than the amplitude $A_2^{(j)}$ at the *lower* interface, in accordance with the readily-established result (cf. Cairns 1979, equation 19)

$$A_1/A_2 = -\Lambda/D_1 = -D_2/\Lambda. \tag{5.1}$$

Since the energy in mode 1 is approximately $\frac{1}{2}\omega^2 k^{-1}(\rho_2+\rho_3)\,|A_2|^2$, it follows from (2.4) that $\omega\partial D/\partial\omega$ must vary roughly as $\exp\,(2kd)$ when $2kd \gg 1$. This is substantiated by the slope of the curve shown in figure 7 (b).

From these results, the time scale of the nonlinear evolution is readily estimated. If the initial amplitudes $|A_1^{(2)}|$, $|A_1^{(3)}|$ at $z = 0$ are of comparable size, say $\mathscr{A}$; if $|A_1^{(1)}|$ is of order $\mathscr{A}\exp\,(-k_1 d)$, giving a mode 1 wave amplitude $|A_2^{(1)}|$ at $z = -d$ also of order $\mathscr{A}$; and if the three respective wavenumbers k_j and frequencies ω_j are of comparable magnitudes, κ and Ω say, then a characteristic time scale for the evolution is

$$\tau \approx \Omega\rho e^{\kappa d}/|\lambda|\kappa\mathscr{A}, \tag{5.2}$$

where ρ is a characteristic density. The factor $\exp\kappa d$ demonstrates that the coupling becomes progressively weaker as κd increases, owing to the fact that two waves are centred at $z = 0$ and the other at $z = -d$.

With typical data $\kappa = 0{\cdot}6\,\mathrm{cm}^{-1}$, $\Omega = 1\,\mathrm{s}^{-1}$, $\rho = 1\,\mathrm{gm\,cm}^{-3}$, $d = 8\,\mathrm{cm}$, and waveslopes $\mathscr{A}\kappa$ of order $O(0{\cdot}05)$, an estimate of τ for $|\lambda|$ of order $O(10^2)\,\mathrm{gm\,cm}^{-3}\,\mathrm{s}^{-2}$ is about 20 s, as compared with a characteristic wave period of order 6 s. This estimate is likely to afford a rough lower bound for τ at such waveslopes, since $|\lambda|$ may be much less than $O(10^2)$ for particular triads (see figure 6). The fact that even for such modest waveslopes, τ can be as short as 20 s, is evidence of a rather strong interaction between the waves. Indeed, the weakly nonlinear theory remains valid only if $\Omega\tau \gg 1$, which in this case implies a restriction that waveslopes $\mathscr{A}\kappa$ be no greater than about $0{\cdot}05$.

6. Discussion

The method outlined by Cairns (1979) yields valuable qualitative results both for linear stability and for three-wave resonance in flows with stepwise or piecewise-linear velocity profiles and step-wise density profiles. Such results follow from separate examination of the waves centred at each fluid interface; and the composite picture emerges from simple energy criteria without need for further detailed analysis of the coupled system.

With the three-wave resonance, an important distinction must be drawn between interactions of the 'explosive' and the 'non-explosive' sort. 'Explosive interactions' occur when the wave of greatest frequency has energy of opposite sign from the other

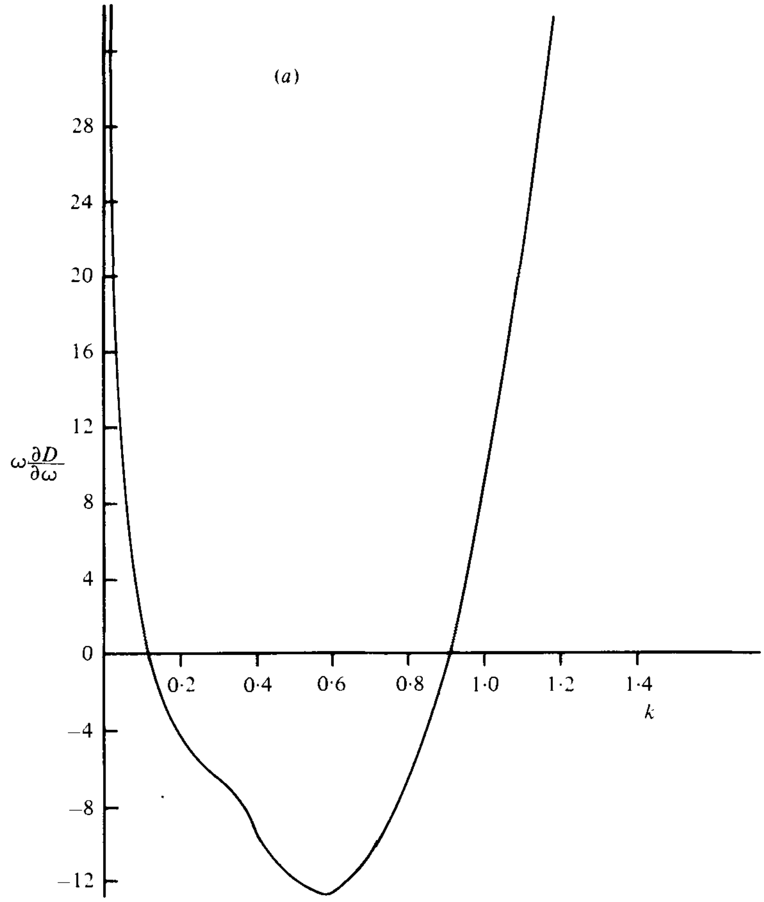

FIGURE 7. For legend see opposite.

two. In such cases, all three waves can grow simultaneously while total wave energy remains constant; and the interaction equations, truncated at second order, have solutions which develop singularities after a finite time. This situation is well-known in plasma physics, but, despite their likely importance, no instances had been recognized in fluid mechanics, until the work of Cairns. However, Craik (1968, 1971) had earlier drawn attention to similar 'explosive' cases in which wave energy is *not* conserved and all three waves grow by extraction of energy from the primary flow through a critical-layer mechanism which is inherently dissipative.

In this paper, we have examined in some detail the linear stability and nonlinear resonance of waves in a three-layer model with two interfaces. Our calculations completely substantiate Cairns' qualitative results, and we have used the insight provided by his results to identify cases which deserved further study. In particular, it was discovered that several types of three-wave resonance could occur; and our prior knowledge enabled selection, for detailed calculation, of those triads which yield explosive instability. On the other hand, the nonlinear analysis leading to results (4.4) holds for *any* triad of two-dimensional waves, irrespective of the respective signs

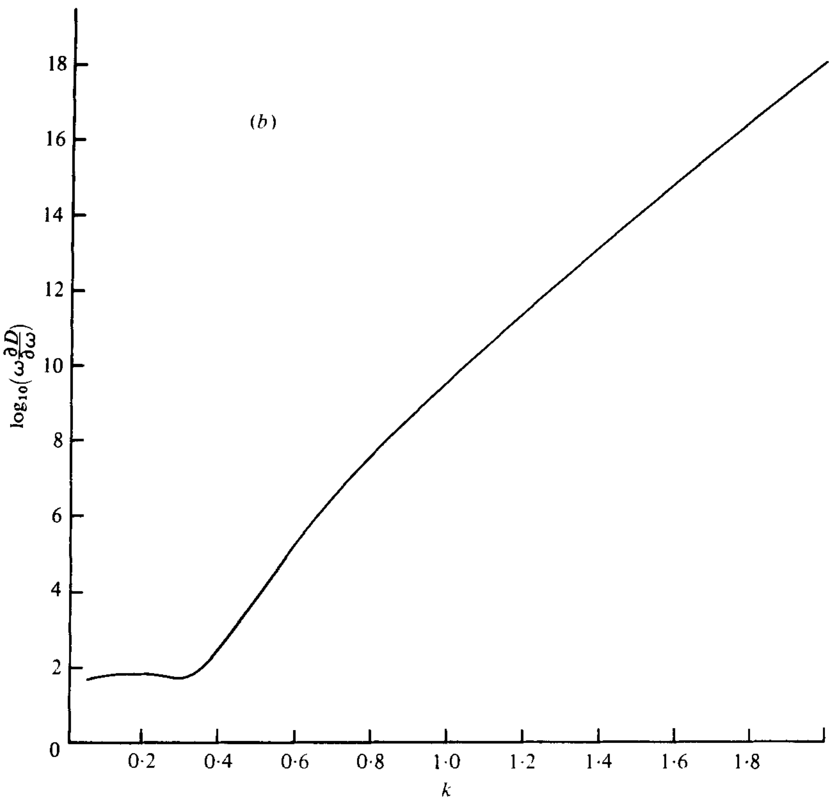

FIGURE 7. (*a*) Graph of $\omega \partial D/\partial \omega$ *vs*. k for mode 3 centred at upper interface; case $U_1 = 10{\cdot}0\,\mathrm{cm\,s^{-1}}$ of figure 5. Note negative energy for $0{\cdot}11 < k < 0{\cdot}91$. (*b*) Graph of $\log_{10}(\omega \partial D/\partial \omega)$ *vs*. k for positive energy mode 1 centred at lower interface; case $U_1 = 10{\cdot}0\,\mathrm{cm\,s^{-1}}$.

of their energies, and represents a substantial generalization of previous work restricted to configurations with just a single interface.

The complexity of our nonlinear analysis, with just two interfaces, indicates the value of Cairns' simple qualitative approach for treating configurations with still more fluid layers. It may be applied, whenever the coupling between interfaces is sufficiently weak, to (inviscid) configurations with any number of layers within which the fluid velocity and density are constant. Piecewise-linear fluid velocities are also permissible, provided the wave motion is entirely two-dimensional, as in Taylor (1931), since the wave motion still remains irrotational in each layer for such flows. But the method requires modification for three-dimensional (oblique) waves; since then, even with linear profiles, the mean vorticity field is distorted by the wave motion and total wave energy need not be conserved when a critical layer is present (for an example, see Craik 1968).

With continuous, curved velocity profiles, the situation is further complicated by the crucial role of the 'critical layer', even in the linear approximation, and it seems unlikely that the qualitative method can be simply modified to deal with such cases. Certainly, Landahl (1962) and Benjamin (1960, 1963) have drawn attention to the phenomenon of linear wave growth caused by damping, which is essentially due to the fact that the wave energy is negative (see Landahl 1962, figure 10 and Cairns 1979, §4); but problems of nonlinear resonant wave triads in dissipative systems with critical layers do not appear to be amenable to such 'short-cuts' (see Usher & Craik 1974, 1975).

The nonlinear analysis given above may be extended to problems in which the wave amplitudes are allowed to depend on the horizontal co-ordinates x, y as well as on time t. With variation in x and t only, the right-hand sides of (4.4) are unchanged, but the terms $dA_1^{(j)}/dt$ on the left must be replaced by $(\partial/\partial t + c_g^{(j)}\,\partial/\partial x)\,A_1^{(j)}$ where $c_g^{(j)} \equiv \partial\omega_j/\partial k_j$ is the group velocity of the jth wave. Extension to cover slow variation of the $A_1^{(j)}$ in x, y and t merely entails calculation of the dispersion relation for oblique waves and inclusion of corresponding y derivatives; but in cases where the respective wavenumber vectors are not nearly collinear, variation in y is not now on a 'long' scale, and the interaction coefficient λ must be recalculated (we do not recommend the latter task!). Exact solutions of such interaction equations, equivalent to (4.4) but with dependence on x, y and t, have recently become known (Zakharov 1976; Craik 1978).

Despite the limitations mentioned above, it seems likely that the possibility of 'explosive' resonant interactions among positive- and negative-energy waves will have significant implications in meteorology and oceanography. Until now, work on resonant interactions among internal waves has been confined to triads (and quartets) in which each member has positive energy. However, the presence of a primary velocity profile which models a wind or current varying with depth may allow the coexistence and 'explosive' interaction of positive- and negative-energy wave-modes. Of course, for internal waves, there is no interfacial surface tension, unlike the situation examined above; so any velocity discontinuity across an interface must exhibit Kelvin–Helmholtz instability to sufficiently short waves (such instability being the manifestation of coalescence of positive- and negative-energy modes). But, even then, resonant interactions among the longer, linearly-stable, wave modes may have an important part to play in the evolution of long wavelength disturbances. An 'explosive' instability of such modes may yield an attractive explanation for the spontaneous generation of internal waves.

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